

Branes for Relativists

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Abstract

Recently, branes in supergravity have become an indispensable tool even for traditional relativists. The purpose of this manuscript is to provide a pedagogical account of the brane technology so that the relativists can use branes in their study. The type IIA supergravity theory is mainly discussed and other cases are briefly mentioned. The repulson singularity is also explained as an interesting application.

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1 Introduction

Let me start to review the framework of the string theory briefly [1, 2, 3]. The perturbative string theory can be formulated as the path-integral of the Poyakov action over all the Euclidean world sheet geometry which has the non-trivial topology characterized by the Euler number

$$\int DX Dh_{ab} \exp\left[-\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu\right], \quad (1)$$

where X^μ and h_{ab} are the 10-dimensional coordinates specifying the string location and the induced metric on the world sheet of the string, respectively. The complicated world sheet topology corresponds to the loop Feynmann diagram in the conventional field theory. In particular, the simplest one, i.e. the sphere, corresponds to the tree diagram. Hence the string classical theory can be obtained by path-integrating the Poyakov action on the sphere. This includes the effects of all of the stringy excitations. In the low energy limit, only the massless states can contribute to the effective action. The stringy corrections can be incorporated systematically like as

$$I = \int d^{10}x \sqrt{-g} e^{-2\phi} \left[\{R + 4(\partial\phi)^2 + \cdots\} + l_s^2 \{R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} + \cdots\} \right], \quad (2)$$

where $l_s = \sqrt{\alpha'}$ is the string length scale and $g = e^\phi$ controls the strength of the string coupling. Notice that the leading term is essentially that of the general relativity. From now on, we shall concentrate ourselves to the analysis of this part. The validity of the analysis is guaranteed if the string coupling constant and the typical energy scale satisfy the inequality $g \ll 1$ and $l_s E \ll 1$. Namely, we consider the low energy classical solution of string theory. However, as the extreme branes are BPS objects, they are related to the non-perturbative objects in string theory, i.e. D-branes. This is the reason why the extreme branes are investigated enthusiastically.

The organization of this review is as follows: In sec.2, the extreme Reissner-Nordstrom black hole solution in general relativity is derived in an appropriate manner for later purpose. In sec.3, general brane solutions are explained. In sec.4, we review the basic branes in type IIA theory. Among them, Dp-branes coupled with R-R gauge fields are important from the string theoretical point of view. In sec.5, we consider the intersecting branes. In sec.6, the repulson singularity is explained. In the final section, we discuss the brane solutions in other theories. In this review, we will not intend to cite the original papers because of the pedagogical nature of this note. Instead, we refer the readers to excellent review works [4, 5, 6, 7].

2 Extreme Reissner-Nordstrom (R-N) in General Relativity

The charged black hole solution is familiar to relativists. Although its extreme limit has less importance in relativity, its stringy counterpart has taken an important role in the

recent development of the nonperturbative aspects of string theory. Especially, the success of the statistical counting of the black hole entropy was impressing. The extreme R-N black hole solution would be a natural starting point for understanding the brane physics. The extreme R-N solution is obtained from the action:

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} [R - \frac{1}{4} F_2^2] \quad (3)$$

where $g_{\mu\nu}$ is the 4-dimensional metric, $F_2 = dA$ is the electro-magnetic field strength. In this setting, it is possible to find a solution for which the test charged particle does not feel any force. The ansatz is

$$ds^2 = -H(r)^{-2} dt^2 + H(r)^2 (dr^2 + r^2 d\Omega^2) \quad , \quad A_t = H(r)^{-1} - 1 \quad , \quad (4)$$

where (0+1)-dimensional Poincare symmetry and the $SO(3)$ -symmetry are imposed. The action for the test brane is defined as

$$S_{particle} = e \int A_\mu dx^\mu - e \int d\tau \quad (5)$$

and the charged particle has the extreme mass, namely $e = m$ in our units. The no force property can be verified by evaluating the above action as

$$S_{particle} = e \int (H^{-1} - 1) - e \int \sqrt{H^{-2} - H^2 v^2} dt \sim e \int H^3 v^2 \quad . \quad (6)$$

Here, the potential terms are canceled out. This type of the solution is sometimes represented by the following table:

	x^0	x^1	x^2	x^3
RN	•	—	—	—

We may call this solution 0-brane, because it has no spatial extension.

Given the metric form, the curvature is easily calculated as

$$R = -2H^{-3} \nabla^2 H \quad . \quad (7)$$

Then, the vacuum Einstein equation reduces to the harmonic equation

$$T^\mu_\mu = 0 \Rightarrow R = 0 \Rightarrow \nabla^2 H = 0 \quad . \quad (8)$$

It is easy to write down the general solution, i.e. multi-black hole solution. Here we consider the simplest solution

$$H = 1 + \frac{Q}{r} \quad . \quad (9)$$

In this case, the metric becomes

$$ds^2 = -(1 + \frac{Q}{r})^{-2} dt^2 + (1 + \frac{Q}{r})^2 (dr^2 + r^2 d\Omega^2) \quad (10)$$

or, by using the coordinate $R = r + Q$,

$$ds^2 = -(1 - \frac{Q}{R})^2 dt^2 + \frac{dR^2}{(1 - \frac{Q}{R})^2} + R^2 d\Omega_2^2 . \quad (11)$$

This is the well known form of the extreme R-N black hole solution.

We have discussed the electric charge only. Because the magnetic charge does not lead to the new metric solution due to the following simple relation

$$F = dA , \quad *F = d\tilde{A} . \quad (12)$$

As we will see, however, the electric solution and the magnetic solution are different in the case of general models.

3 Branes:general

Here, we would like to generalize the extreme Reissner-Nortstrom solution to the higher dimensional gravity with the dilaton. One new thing comes here. The higher rank gauge fields are allowed to couple with the dilaton.

The action is

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \{ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \frac{1}{(p+2)!} e^{a\phi} F_{p+2}^2 \} \quad (13)$$

where $g_{\mu\nu}$ is the D -dimensional metric, ϕ is the dilaton, and $F_{p+2} = dA_{p+1}$ is the $(p+2)$ -form field strength. The coupling constant a is a key parameter in the brane physics. It reflects the origin of the fields, namely NS-NS or R-R, and the rank of the tensor field. In order to find the brane solutions, we need to impose the $(p+1)$ -dimensional Poincare invariance and $SO(d)$ -symmetry of the transverse space to the brane. Here, d is the co-dimensions of the brane. Moreover, as we seek the extreme solution, we need to consider the no force condition. The action of the test brane is given by

$$S_{brane} = \mu_p \int A_{p+1} - \tau_p \int d^{p+1} x f(\phi) \sqrt{-\det G_{ab}} , \quad (14)$$

where μ_p and τ_p are the charge and the tension of the brane and G_{ab} is the induced metric on the brane. The extreme solution is defined by $\mu_p = \tau_p$. Here, $f(\phi)$ takes the form

$$f(\phi) = e^{\frac{p-3}{4}\phi} \quad (15)$$

for the R-R forms and

$$f(\phi) = e^{\frac{p+1}{4}\phi} \quad (16)$$

for NS-NS forms in the type II supergravity theories. In the case of the M-theory, as there exists no dilaton, this part of the action is trivial. The ansatz we imposed is

$$ds^2 = H^{-2\frac{d-2}{\Delta}} \{-dt^2 + dy_1^2 + \cdots + dy_p^2\} + H^{2\frac{p+1}{\Delta}} \{dr^2 + r^2 d\Omega_{d-1}^2\} , \quad (17)$$

$$e^\phi = H^{\frac{a(D-2)}{\Delta}}, \quad (18)$$

$$A_{ty_1 \dots y_p} = \sqrt{\frac{2(D-2)}{\Delta}} (H^{-1} - 1), \quad (19)$$

$$\Delta = (p+1)(d-2) + \frac{1}{2}a^2(D-2), \quad (20)$$

where H depends only on r and $D = p + d + 1$. It is easy to verify the no force property of the brane for each model separately. We leave it as an exercise for the reader. After the straightforward but tedious calculation, we have the harmonic equation

$$\nabla^2 H = 0. \quad (21)$$

Its solution is given by

$$H = 1 + \frac{h^{d-2}}{r^{d-2}} \quad (22)$$

where the integration constant h is related to the charge of the brane. So far, we have discussed the electric solution. The following relation suggests the property of the magnetic solution,

$$F_{p+2} = dA_{p+1}, \quad {}^*F_{D-p-2} = dA_{D-p-3} \quad (23)$$

Namely, the magnetic solution is nothing but $D - p - 4$ brane solution.

4 Basic branes in type IIA SUGRA

The low energy effective action of the type IIA superstring theory is the type IIA supergravity theory. The bosonic part of the action in the string frame is given by

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g_s} \{ e^{-2\phi} \left[R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_3^2 \right] - \frac{1}{4} F_2^2 - \frac{1}{48} F_4^2 \} \quad (24)$$

The transformation rule

$$g_{\mu\nu}^s = e^{\frac{\phi}{2}} g_{\mu\nu}^E \quad (25)$$

gives the action in the Einstein frame

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g_E} \{ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{-\phi} H_3^2 - \frac{1}{4} e^{\frac{3}{2}\phi} F_2^2 - \frac{1}{48} e^{\frac{1}{2}\phi} F_4^2 \}. \quad (26)$$

Now, we can use the results of the previous section in order to obtain the brane solutions. In this section, we consider the single brane, and the intersecting branes are the subjects of the next section.

4.1 NS-NS sector

The gauge field in the NS-NS sector is

$$H_3 = dB_2 . \quad (27)$$

The antisymmetric tensor field B_2 naturally couples with the fundamental string. From the action one can read off $a = -1$ and then obtain $\Delta = 2 \cdot 6 + 1 \cdot 8/2 = 16$. Notice that $D = 10$ in superstring theories. The electric brane is the fundamental 1-brane:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
F1	•	•	—	—	—	—	—	—	—	—

The solution is given by

$$ds_{F1}^2 = H^{-\frac{3}{4}}(-dt^2 + dy_1^2) + H^{\frac{1}{4}}(dx_1^2 + \cdots + dx_8^2) , \quad (28)$$

$$e^\phi = H^{-\frac{1}{2}} , \quad (29)$$

$$B_{ty_1} = H^{-1} - 1 , \quad (30)$$

$$H = 1 + \frac{h^6}{r^6} . \quad (31)$$

It is easy to obtain the metric in the string frame:

$$d\hat{s}_{F1}^2 = H^{-1}(-dt^2 + dy_1^2) + dx_1^2 + \cdots + dx_8^2 . \quad (32)$$

The magnetic brane is NS 5-brane:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	•	•	•	•	•	•	—	—	—	—

The metric is given by

$$ds_{NS5}^2 = H^{-\frac{1}{4}}(-dt^2 + dy_1^2 + \cdots + dy_5^2) + H^{\frac{3}{4}}(dx_1^2 + \cdots + dx_4^2) , \quad (33)$$

$$e^\phi = H^{\frac{1}{2}} , \quad (34)$$

$$H_{\theta_1, \theta_2, \theta_3} = Q\omega_3 , \quad (35)$$

$$H = 1 + \frac{h^2}{r^2} . \quad (36)$$

Here, Q is related to the other constant h . In the string frame, the metric becomes

$$d\hat{s}_{NS5}^2 = -dt^2 + dy_1^2 + \cdots + dy_5^2 + H(dx_1^2 + \cdots + dx_4^2) . \quad (37)$$

4.2 R-R sector

The gauge fields in the R-R sector are

$$F_2 = dC_1 \quad (38)$$

and

$$F_4 = dC_3 . \quad (39)$$

From the action, we can find $a_p = (3-p)/2$ and then $\Delta = (p+1)(7-p) + (3-p)^2 = 16$. Then, the solution becomes

$$ds_{Dp}^2 = H^{-\frac{7-p}{8}}(-dt^2 + dy_1^2 + \cdots + dy_p^2) + H^{\frac{p+1}{8}}(dx_1^2 + \cdots + dx_{9-p}^2) , \quad (40)$$

$$e^\phi = H^{\frac{3-p}{4}} , \quad (41)$$

$$C_{ty_1 \cdots y_p} = H^{-1} - 1 , \quad (42)$$

$$H = 1 + \frac{h^{7-p}}{r^{7-p}} . \quad (43)$$

Also, the metric in the string frame is

$$d\hat{s}_{Dp}^2 = H^{-\frac{1}{2}}(-dt^2 + dy_1^2 + \cdots + dy_p^2) + H^{\frac{1}{2}}(dx_1^2 + \cdots + dx_{9-p}^2) . \quad (44)$$

5 Harmonic Superposition

Interestingly, based on the previous results, one can derive various solutions without calculations. The superposition rule is the only necessary ingredient for this purpose. For example, in the case of the D-brane, we obtain the following superposition rules

$$Dp \cap Dp = p - 2 , \quad D(p-2) \cap Dp = p - 3 , \quad D(p-4) \cap Dp = p - 4 , \quad (45)$$

where the right hand side represents the number of the common spatial dimensions. These rules can be deduced from the consideration of the supersymmetry. The extreme brane preserves one half of the space-time supersymmetry of the theory. So the problem is how to combine the branes so as not to break all of the supersymmetry of the theory. In the case of the Dp-branes, we obtained the above rules. Of course, these rules can be obtained by the elementary analysis of the equations of motion.

As an illustration, we consider the D6-D2 system:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D6	•	•	•	•	•	•	•	—	—	—
D2	•	•	•	—	—	—	—	—	—	—

As $D6 \cap D2 = 2$, the 1, 2-directions are chosen as the common directions. The solution is obtained by the simple superposition of both solutions

$$d\hat{s}_{Dp}^2 = H_2^{-\frac{1}{2}} H_6^{-\frac{1}{2}} (-dt^2 + dy_1^2 + dy_2^2) + H_2^{\frac{1}{2}} H_6^{-\frac{1}{2}} (dy_3^2 + dy_4^2 + dy_5^2 + dy_6^2) + H_2^{\frac{1}{2}} H_6^{\frac{1}{2}} (dr^2 + r^2 d\Omega_2^2) \quad (46)$$

where

$$H_2 = 1 + \frac{r_2}{r}, \quad H_6 = 1 + \frac{r_6}{r}. \quad (47)$$

Notice that the power of the harmonics is determined by the dimensions of the transverse spaces to all of the branes. This is the so-called smearing effects.

6 Repulson Singularity

It would be interesting to study the wrapped brane $D6 - D2^*$, because it has a naked time-like singularity and its resolution in the string theoretical context is known. Notice that the four-dimensional manifold K3 is supersymmetric itself, hence the wrapped D6 brane solution on K3 is also supersymmetric. The resulting solution induces negative charged D2 branes. Then, this system is often called $D6 - D2^*$. The metric is given by

$$d\hat{s}_{Dp}^2 = H_2^{-\frac{1}{2}} H_6^{-\frac{1}{2}} (-dt^2 + dy_1^2 + dy_2^2) + H_2^{\frac{1}{2}} H_6^{\frac{1}{2}} (dr^2 + r^2 d\Omega_2^2) + V^{\frac{1}{2}} H_2^{\frac{1}{2}} H_6^{-\frac{1}{2}} ds_{K3}^2, \quad (48)$$

where

$$H_2 = 1 - \frac{r_2}{r}, \quad H_6 = 1 + \frac{r_6}{r}. \quad (49)$$

Here, ds_{K3}^2 is the metric of a K3 surface of unit volume. V is the volume of the K3 as measured at infinity, but the supergravity solution adjusts itself such that $V(r) = V H_2 H_6^{-1}$ is the measured volume of K3 at radius r . Other fields such as R-R forms can be obtained easily. Considering the connection with the string theory, we obtain

$$r_2 = \frac{(2\pi)^4 g N \alpha^{\frac{5}{2}}}{2V}, \quad r_6 = \frac{8N \alpha^{\frac{4}{2}}}{2}. \quad (50)$$

There is a time-like naked singularity at $r = r_2$, known as “repulson” (see Fig.1). The curvature diverges there which is related to the fact that the volume of the K3 goes to zero there.

Let us see the repulsive nature of this solution. The Newtonian potential is given by

$$\Phi = \frac{1}{2} [-g_{00} - 1] = \frac{1}{2} \left[\left(1 - \frac{r_2}{r}\right)^{-\frac{1}{2}} \left(1 + \frac{r_6}{r}\right)^{-\frac{1}{2}} - 1 \right] \quad (51)$$

Let $r \rightarrow \infty$, then

$$\Phi \approx \frac{r_2 - r_6}{4r} \approx -\frac{g N \alpha^{\frac{4}{2}}}{8r} \left[1 - \frac{V^*}{V} \right] \quad (52)$$

thus, at the infinity, the gravity force is attractive as is expected. Here we assumed

$$V^* = (2\pi)^4 \alpha'^2 < V. \quad (53)$$

As you can see, however, the potential becomes repulsive near the naked singularity. The resolution of this singularity will be discussed in the review talk by Yamaguchi.

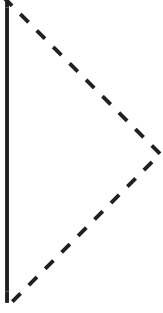


Figure 1: Penrose diagram for $D6 - D2^*$ solution. The bold line represents the time-like naked singularity.

7 Others

The method explained in this review can be easily extended to the other theories.

In the case of 11 dimensional supergravity, the bosonic part of the action is given by

$$S = \frac{1}{16\pi G_{10}} \int d^{11}x \sqrt{-g} \{ R - \frac{1}{48} H_4^2 \} , \quad (54)$$

where no dilaton exists. The possible solutions are

$$W , KK , M2 , M5 . \quad (55)$$

The Kalza-Klein wave solution is

$$\begin{aligned} ds_W^2 &= -dt^2 + dy^2 + K(dt - dy)^2 + dx_1^2 + \dots + dx_9^2 , \\ K &= \frac{Q}{r^7} . \end{aligned} \quad (56)$$

The Kalza-Klein monopole solution is

$$\begin{aligned} ds_{KK}^2 &= -dt^2 + dy_1^2 + \dots + dy_6^2 + H^{-1}(dz + A_i dx^i)^2 + H dx^i dx_i , i = 1...3 , \\ H &= 1 + \frac{Q}{r} , \quad \partial_i A_j - \partial_j A_i = -\epsilon_{ijk} \partial_k H . \end{aligned} \quad (57)$$

Both of the above solutions have off-diagonal components, then they do not belong to the category discussed in the previous sections. However, they can be derived by reversing the Kalza-Klein procedure to derive 11-dimensional metric from the 10-dimensional brane solution.

The electric 2-brane for the 3-form field can be obtained by setting $a = 0$ in the general brane solution. The result is

$$\begin{aligned} ds_{M2}^2 &= H^{-\frac{2}{3}}(-dt^2 + dy_1^2 + dy_2^2) + H^{\frac{1}{3}}(dx_1^2 + \dots + dx_8^2) , \\ H &= 1 + \frac{Q}{r^6} . \end{aligned} \quad (58)$$

Similarly, the magnetic 5-brane is deduced as

$$\begin{aligned} ds_{M5}^2 &= H^{-\frac{1}{3}}(-dt^2 + dy_1^2 + \cdots + dy_5^2) + H^{\frac{2}{3}}(dx_1^2 + \cdots + dx_5^2) , \\ H &= 1 + \frac{Q}{r^3} . \end{aligned} \quad (59)$$

In the case of the type IIB supergravity, the bosonic part of action in the string frame is

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g_s} \{ e^{-2\phi} \left[R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_3^2 \right] - \frac{1}{2} F_0^2 - \frac{1}{12} F_3^2 - \frac{1}{240} F_5^2 \} \quad (60)$$

and the action in the Einstein frame is

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g_E} \{ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{-\phi} H_3^2 - \frac{1}{2} e^{2\phi} F_0^2 - \frac{1}{12} e^\phi F_3^2 - \frac{1}{240} F_5^2 \} . \quad (61)$$

In this case, the possible branes are

$$W , KK , F1 , NS5 , D(-1) , D1 , D3 , D5 . \quad (62)$$

Compare with the case of the type IIA supergravity

$$W , KK , F1 , NS5 , D0 , D2 , D4 , D6 . \quad (63)$$

As the formula for the metric in type IIB theory is almost the same as those in type IIA theory, we do not display them here.

References

- [1] J. Polchinski, “String Theory” , Vols. I and II (Cambridge University Press, 1998).
- [2] M.B. Green, J.H. Schwarz and E. Witten, “Superstring Theory”, Vols.1 and 2; Cambridge Univ. Pr. (1987) (Cambridge Monographs on Mathematical Physics).
- [3] E. Kiritsis, “Introduction to Superstring Theory”, hep-th/9709062.
- [4] D. Marolf, String/M-branes for Relativists, gr-qc/9908045.
- [5] A. Peet, TASI Lectures on Black Holes in String Theory, hep-th/0008241.
- [6] R. Argurio, Brane Physics in M-theory, hep-th/9807171.
- [7] K.S. Stelle, Lectures on Supergravity p-branes, hep-th/9701088.